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THE MOTION OF A PROJECTILE IN A MEDIUM RESISTING AS THE CUBE OF THE VELOCITY.

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For the motion of a projectile in a medium resisting as the *square* of the velocity, works on Mechanics, after assuming the fundamental equations of translatory motion,

$$\left|\begin{array}{l} \Sigma P\cos\alpha-\Sigma m(d^2x/dt^2)=0\ldots\ldots(\alpha), \\ \Sigma P\cos\beta-\Sigma m(d^2y/dt^2)=0\ldots\ldots(\beta), \\ \Sigma P\cos\gamma-\Sigma m(d^2z/dt^2)=0\ldots\ldots(\gamma), \end{array}\right| \ldots\ldots(1),$$

easily deduce the following equations:

$$\left|\begin{array}{l} M(d^2x/dt^2)=Mg\cos\alpha+Mcv^2\cos\alpha'\ldots\ldots(\alpha), \\ M(d^2y/dt^2)=Mg\cos\beta+Mcv^2\cos\beta'\ldots\ldots(\beta), \\ M(d^2z/dt^2)=Mg\cos\gamma+Mcv^2\cos\gamma'\ldots\ldots(\gamma), \end{array}\right| \ldots\ldots(2).$$

Let w be the *terminal* velocity of the projectile in the medium, and W the weight of the projectile in pounds; then the resistance of the air, when the velocity of the projectile is v, is equivalent to a force of $R=(v/w)^3W$ pounds. At the same time, the gravitational retardation produced is equivalent to a force $G=(v/w)^3g$ pounds.

Refer the motion of the projectile to oblique co-ordinate axes. Let the axis OX extend in the direction of projection at the point of infinite velocity, and the axis OY extend vertically downwards. In other words, cause the trajectory to lie in the vertical *co-ordinate-plane* XOY. This will enable us practically to *ignore* the co-ordinate axis OZ, and the equation (γ) in (1) and (2).

In the case under consideration, $\cos\alpha=0$, $\cos\beta=-1$, $\cos\gamma=0$, $\cos\alpha'=-(dx/ds)$, $\cos\beta'=-(dy/ds)$, $\cos\gamma'=(dz/ds)$, and $v=ds/dt$.

Transforming (2) under these conditions,

$$\left|\begin{aligned}
\frac{d^2x}{dt^2} &= -\frac{g}{w^3}\left(\frac{ds}{dt}\right)^3\frac{dx}{ds}\ldots\ldots(\alpha),\\
\frac{d^2y}{dt^2} &= -\frac{g}{w^3}\left(\frac{ds}{dt}\right)^3\frac{dy}{ds}+g\ldots\ldots(\beta),\\
\frac{d^2z}{dt^2} &= -\frac{g}{w^3}\left(\frac{ds}{dt}\right)^3\frac{dz}{ds}\ldots\ldots(\gamma),
\end{aligned}\right|\ldots\ldots(3).$$

From (β) and (α) in (3),

$$\left(\frac{ds}{dt}\right)^3 = -\frac{w^3}{g}\left(\frac{d^2y}{dt^2}-g\right)\frac{ds}{dy} = -\frac{w^3}{g}\left(\frac{d^2x}{dt^2}\right)\frac{ds}{dx}\ldots\ldots(a).$$

Dividing (a) by ds/dt, etc.,

$$\left(\frac{d^2y}{dt^2}\right)\frac{dt}{dy}-\left(\frac{d^2x}{dt^2}\right)\frac{dt}{dx}=g\frac{dt}{dy}\ldots\ldots(b).$$

Multiplying (b) by $(dx/dt)(dy/dt)$, etc.,

$$\frac{d^2y}{dt^2}\cdot\frac{dx}{dt}-\frac{d^2x}{dt^2}\cdot\frac{dy}{dt}=g\frac{dx}{dt}\ldots\ldots(c).$$

$$\therefore\left(\frac{d^2y}{dt^2}\cdot\frac{dx}{dt}-\frac{d^2x}{dt^2}\cdot\frac{dy}{dt}\right)\frac{dt}{dx}\bigg/\left(\frac{dx}{dt}\right)^2=g\frac{dx}{dt}\bigg/\left(\frac{dx}{dt}\right)^2\ldots\ldots(d).$$

$$\therefore\frac{d}{dt}\left(\frac{dy}{dt}\bigg/\frac{dx}{dt}\right)\frac{dt}{dx}=g\left(1\bigg/\frac{dx}{dt}\right)\ldots\ldots(e).$$

$$\therefore\frac{dp}{dt}=g\frac{dt}{dx},\ \text{or}\ \frac{dp}{dt}\cdot\frac{dx}{dt}=g\ldots\ldots(f),$$

in which p stands for dy/dx. *Apparently* (f) is *independent* of the law of resistance of the medium in the direction opposite to motion.

Assuming that the co-ordinate axis OX makes an angle ω with a horizontal plane, we have from the parallelogram of *velocity-components*

$$\cos(90-\omega)=\frac{(dy/dt)^2+(dx/dt)^2-(ds/dt)^2}{2(dy/dt)(dx/dt)}\ldots\ldots(g).$$

$$\therefore\left(\frac{ds}{dt}\right)^2=\left(\frac{dy}{dt}\right)^2-2\left(\frac{dy}{dt}\right)\left(\frac{dx}{dt}\right)\sin\omega+\left(\frac{dx}{dt}\right)^2$$

$$=\left(\frac{dx}{dt}\right)^2\left[\left(\frac{dy}{dx}\right)^2-2\left(\frac{dy}{dx}\right)\sin\omega+1\right]=\left(\frac{dx}{dt}\right)^2[p^2-2p\sin\omega+1]:\ldots\ldots(h).$$

From (a) in (3), by means of (h), may be deduced

$$\frac{d^2x}{dt^2}=-\frac{g}{w^3}\left(\frac{ds}{dt}\right)^2\left(\frac{ds}{dt}\times\frac{dx}{ds}\right)=-\frac{g}{w^3}\left(\frac{ds}{dt}\right)^2\frac{dx}{dt}$$

$$=-\frac{g}{w^3}\left(\frac{dx}{dt}\right)[p^2-2p\sin\omega+1]\ldots\ldots(i).$$

Eliminating g from (i), by means of (f),

$$\frac{dx^2}{dt^2}=-\frac{1}{w^3}\left(\frac{dp}{dt}\cdot\frac{dx}{dt}\right)\left(\frac{dx}{dt}\right)^3\left[\left(\frac{dy}{dx}\right)^2-2\left(\frac{dy}{dx}\right)\sin\omega+1\right]\ldots.(j).$$

$$\therefore\ w^2\left(\frac{dx}{dt}\right)^{-4}\frac{d^2x}{dt^2}=-\left[\left(\frac{dy}{dx}\right)^2-2\left(\frac{dy}{dx}\right)\sin\omega+1\right]\frac{d}{dt}\left(\frac{dy}{dx}\right)\ldots.(k).$$

Remembering that when $dy/dx=0$, $dx/dt=\infty$, and integrating (k),

$$\tfrac{1}{3}w^3(dx/dt)^{-3}=\tfrac{1}{3}p^3-p^2\sin\omega+p\ldots.(l).$$

$$\therefore\ dx/dt=w/\sqrt[3]{(p^3-3p^2\sin\omega+3p)}\ldots.(m).$$

Obvious transformations of (f) give

$$\frac{dp}{dt}=g\left(\frac{dx}{dt}\right)^{-1},\text{ or }\frac{dp}{dx}\times\frac{dt}{dx}=g\left(\frac{dx}{dt}\right)^{-2}\ldots.(a').$$

$$\therefore\ dp/dx=(g/w^2)[p^3-3p^2\sin\omega+3p]^{\frac{2}{3}}\ldots.(n).$$

Taking reciprocals of (n), etc.,

$$(g/w^2)(dx/dp)=1/[p^3-3p^2\sin\omega+3p]^{\frac{2}{3}} \ldots.(o).$$

Multiplying the left-hand member of (o) by dy/dx and the right-hand member by p, etc.,

$$(g/w^2)(dy/dp)=p/[p^3-3p^2\sin\omega+3p]^{\frac{2}{3}} \ldots.(p).$$

From (f) and (m), can easily be deduced

$$\frac{dp}{dt}=g\Big/\frac{dx}{dt}=(g/w)[p^3-3p^2\sin\omega+3p]^{\frac{1}{3}} \ldots.(q).$$

Also, from (o), (p), and (q), respectively, by obvious integrations, etc.,

$$\frac{gx}{w^2}=\int_0^\infty d\left(\frac{dy}{dx}\right)\Big/\left[\left(\frac{dy}{dx}\right)^3-3\left(\frac{dy}{dx}\right)^2\sin\omega+3\left(\frac{dy}{dx}\right)\right]^{\frac{2}{3}} \ldots.(r),$$

$$\frac{gy}{w^2}=\int_0^\infty \frac{dy}{dx}d\left(\frac{dy}{dx}\right)\Big/\left(\frac{dy}{dx}\right)^3-3\left(\frac{dy}{dx}\right)^2\sin\omega+3\left(\frac{dy}{dx}\right)\Big]^{\frac{2}{3}} \ldots.(s),$$

$$\frac{gt}{w}=\int_0^\infty d\left(\frac{dy}{dx}\right)\Big/\left[\left(\frac{dy}{dx}\right)^3-3\left(\frac{dy}{dx}\right)^2\sin\omega+3\left(\frac{dy}{dx}\right)\right]^{\frac{1}{3}} \ldots.(t).$$

Assume $z=M^2\sqrt[3]{(p^3-3p^2\sin\omega+3p)/p} \ldots.(\beta')$.

Cubing (β'), differentiating, etc.,

$$6z^2dz=6M^6[(p\sin\omega-2)/p^3]dp \ldots.(\gamma').$$

From (β'), after cubing, can be formed the expression,

$$4z^3-g_3=[(4M^6-g_3)p^2-12M^6p\sin\omega+12M^6]/p^2 \ldots.(\delta'),$$

which is a perfect square when we have $4M^6-g_3=3M^6\sin^2\omega$, or when $g_3=M^6(4-3\sin^2\omega)$. Substituting this value of g_3 in (δ), etc.,

$$\sqrt{(4z^3-g_3}=M^3(p\sin\omega-2)\sqrt{(3)}/p \ldots.(\epsilon').$$

Dividing (γ') by (ε'), etc., also making the *arbitrary* constant factor $M^2=\frac{1}{3}$, we have

$$\frac{6z^2dz}{\sqrt{(4z^3-g_3)}}=\frac{6M^6(p\sin\omega-2)dp}{p^3}\times\frac{p}{M^3(p\sin\omega-2)\sqrt{3}} \ldots.(\varsigma').$$

$$\therefore \frac{dz}{\surd(4z^3-g_3)}=\frac{M^3\surd(3)}{3p^2z^2}\times\frac{g[p^3-3p^2\sin\omega+3p]^{\frac{3}{3}}\,dx}{w^2}$$

$$=\frac{M^3\surd(3)\times p^2}{3p^2\times M^4[p^3-3p^2\sin\omega+3p]^{\frac{2}{3}}}\times\frac{g[p^3-3p^2\sin\omega+3p]^{\frac{2}{3}}\,dx}{w^2}\ldots.(\eta').$$

$$\therefore \frac{gx}{w^2}=\int_0^\infty\frac{dz}{\surd(4z^3-g_3)},={}^*(p,\,u)^{-1}(z;\,0,\,g_3)\ldots.(\theta').$$

That is, $z={}^*(p,\,u)\left(\frac{gx}{w^2};\,0,\,g_3\right)\ldots.(\iota')$,

which is an *incomplete* form of the *first* of the three Weierstrassian functions by means of which it is possible to express any elliptic integral; and this function is expressed in its *inverted* form according to the Abelian spstem of notation. Now, ${}^*(p,\,u)'\left(\frac{gx}{w^2}\right)=\frac{p\sin\omega-2}{3p}$; and if $z=a$ at the vertical asymptote, where $p=\infty$, we must have

$${}^*(p,\,u)'\left(\frac{ga}{w^2}\right)=\frac{\sin\omega}{3},\text{ and }{}^*(p,\,u)\left(\frac{ga}{w^2}\right)=\tfrac{1}{3}.$$

$$\therefore\ {}^*(p,\,u)'\left(\frac{ga}{w^2}\right)-{}^*(p,\,u)'\left(\frac{gx}{w^2}\right)=\frac{2}{3p}\ldots.(\kappa').$$

From (κ') by obvious transformations

$$p=\frac{dy}{dx}=\tfrac{2}{3}/\left[{}^*(p,\,u)'\left(\frac{ga}{w^2}\right)-{}^*(p,\,u)'\left(\frac{gx}{w^2}\right)\right]\ldots.(\lambda').$$

$$\therefore\ y=\int_0\left[6{}^*(p,\,u)^2\left(\frac{ga}{w^2}\right)/\left[{}^*(p,\,u)'\left(\frac{ga}{w^2}\right)-{}^*(p\text{-}\,u)'\left(\frac{gx}{w^2}\right)\right]\right]dx\ldots.(\mu'),$$

which is the equation of the trajectory expressed according to the Weierstrassian system of functional notation.

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*Being without the proper character for the $(p,\,u)$ function we designate it by $(p,\,u)$.